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THE PROBLEM OF THE EXISTENCE OF OPTIMAL TRAJECTORIES OF NONLINEAR SYSTEMS

[Following is a translation of an article by F. M. Kirillova entitled "K Probleme Sushchestvovaniya optimal'nykh trayektoriy Nelineynykh Sistem" (English version above), in Izvestiya Vysshikh Uchebnykh Zavedeniy Matematika (Mathematical News of Higher Educational Institutions), No 2, Kazan, Mar-Apr 61, pp 41-53.]

Questions of the existence of optimal controls (under certain restrictions) for the case of a transient process described by a linear system of differential equations have been examined in detail in works [1-5].

In the present article under the supposition that the control system is given by non-linear differential equations, a theorem on the existence of a solution of the optimal control problem will be proved. This problem was proposed by N. N. Krasovskiy and completed under his direction.

§1. Let a differential equation be given

$$\frac{dx}{dt} = f(x, t) + B(t)u(t), \qquad (1.1)$$

where $x = (x_1(t), \ldots, x_n(t))$ image vector, $B(t)$ -- a matrix whose elements $b_{ij}(t)$, $i = 1, \ldots, n$, $j = 1, \ldots, r$ are continuous in time t, $u(t) = (u_1(t), \ldots, u_r(t))$ -- controlling vector. As in [1-5] we consider that the vector $u(t)$ is piecewise continuous and satisfies the inequality

$$\max |u_j(t)| \leqslant N^{1)}, \qquad j = 1, \ldots, r. \qquad (1.2)$$

1) Strictly speaking, we require the inequality $\max |u_j(t)| \leqslant N$ not on the entire segment $t_0 \leqslant t \leqslant T$; the relation $\max |u_j(t)| > N$ is possible, but on distinct isolation points we will ignore it.

N -- constant.

The function $f(x, t) = (f_1(x_1, \ldots, x_n, t), \ldots, f_n(x_1, \ldots, x_n, t))$ is continuous in t, and possesses continuous, bounded partial derivatives with respect to $x_1, \ldots, x_n$, $\left|\frac{\partial f_i}{\partial x_j}\right| \leq L$ (L -- constant) and $f(0, t) = 0$.

The optimal control problem consists in the following: let at a moment of time t_0 there be a point with coordinates $x(t_0) = x_0$ which is traveling along the trajectory of equation (1.1); it is necessary to select a controlling vector such that the point reaches the origins of the coordinates in the shortest time T. In this case $u(t)$ is called the optimal controlling vector, and the time T the optimal control time.

In §3 of this article it will be shown that if for (1.1) with some control $u(t)$ which satisfies condition (1.2), the coordinate origin is accessible, there exists at least one optimal control $u^0(t)$, also satisfying condition (1.2) and being piecewise constant. The theorem is proved under certain limitations, imposed upon a system of a linear approximation for (1.1).

We suppose that there exists a sequence of control vectors $u^{(k)}(t)$ of the system (1.1), the corresponding trajectories of $x(x_0, t_0, u^{(k)}(t), t)$ of which satisfy the relations

$$x(x_0, t_0, u^{(k)}(t), t_0 + T_k) = 0, \qquad k = 1, 2, \ldots,$$

where $T_k > T_{k+1}$ and $\lim T_k = T > 0$ for $k \to \infty$, and the same vectors satisfy the condition (1.2). Let there not exist a control $u(t)$ which would satisfy condition (1.2) and the inequality

$$x(x_0, t_0, u(t), t_0 + \theta) = 0 \text{ for } \theta < T.$$

Henceforth such a sequence of control vectors $u^{(k)}(t)$ will be called a minimizing sequence. Our problem consists in proving that if there exists a control $u^{(1)}(t)$, for which the origin of the coordinates is accessible in some time T_1, then (under certain restrictions) there exists a piecewise

constant optimal vector $u^0(t)$ such that along a corresponding trajectory $x(x_0, t_0, u^0(t), t)$ the origin of the coordinates is reached in time T. At first we will prove the correctness of lemmas important for the subsequent developments.

Lemma 1.1. If $u^{(1)}(t)$, $u^{(2)}(t)$, $t_0 \leqslant t \leqslant \tau$ -- control vectors of the equation (1.1), then for corresponding solutions $x^{(1)}(t) = x(x_0, t_0, u^{(1)}(t), t)$ $x^{(2)}(t) = x(x_0, t_0, u^{(2)}(t), t)$ an accurate estimate is

$$\sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| \leqslant rnM \int_{t_0}^{t} \sum_{j=1}^{r} |u_j^{(1)}(t) - u_j^{(2)}(t)| \, dt e^{nL(t-t_0)},$$

where $M = \max |b_{ij}(t)|$, $t_0 \leqslant t \leqslant \tau$.

Proof. Seeing that

$$x^{(1)}(t) = x_0 + \int_{t_0}^{t} [f(x^{(1)}, s) + B(s) u^{(1)}(s)] \, ds,$$

$$x^{(2)}(t) = x_0 + \int_{t_0}^{t} [f(x^{(2)}, s) + B(s) u^{(2)}(s)] \, ds,$$

then, subtracting the second equation from the first, we obtain

$$x^{(1)}(t) - x^{(2)}(t) = \\ = \int_{t_0}^{t} [f(x^{(1)}, s) - f(x^{(2)}, s) + B(s)(u^{(1)}(s) - u^{(2)}(s))] \, ds.$$

But according to the condition, $\left|\frac{\partial f_i}{\partial x_j}\right| \leqslant L$, it follows that

$$|f_i(x^{(1)}, t) - f_i(x^{(2)}, t)| \leqslant L \sum_{k=1}^{n} |x_k^{(1)}(t) - x_k^{(2)}(t)|.$$

Thus, we have

$$|x_i^{(1)}(t) - x_i^{(2)}(t)| \leqslant \int_{t_0}^{t} \Big[L \sum_{k=1}^{n} |x_k^{(1)}(s) - x_k^{(2)}(s)| +$$

$$+ \sum_{j=1}^{r} |b_{ij}(s)| \, |u_j^{(1)}(s) - u_j^{(2)}(s)| \Big] ds.$$

We sum this inequality on i

$$\sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| \leqslant Ln \int_{t_0}^{t} \sum_{k=1}^{n} |x_k^{(1)}(s) - x_k^{(2)}(s)| ds +$$

$$+ \max |b_{ij}(t)| \, rn \int_{t_0}^{t} \sum_{j=1}^{r} |u_j^{(1)}(s) - u_j^{(2)}(s)| ds.$$

Signifying $\max |b_{ij}(t)|$, $t_0 \leqslant t \leqslant \tau$, by M and applying lemma [7] we obtain the desired inequality

$$\sum_{i=1}^{n} |x_i^{(1)}(t) - x_i^{(2)}(t)| \leqslant rnM \int_{t_0}^{t} \sum_{j=1}^{r} |u_j^{(1)}(s) - u_j^{(2)}(s)| ds \, e^{nL(t-t_0)}.$$

From such an inequality, it easily follows that if

$$\operatorname{mes} E(|u_j^{(1)}(t) - u_j^{(k)}(t)| \geqslant \sigma) \to 0^{1)}$$

for each $j = 1, 2, \ldots, r$, $\sigma > 0$ and $k \to \infty$, then $x_i^{(k)}(t) \to x_i^{(1)}(t)$ for $k \to \infty$.

Lemma 1.2. If $u^{(k)}(t)$ -- minimizing sequence of the equation (1.1), then the sequence of trajectories $x(x_0, t_0, u^{(k)}(t), t)$ contains at least one uniformly convergent subsequence $x^{(k_p)}(t) = x(x_0, t_0, u^{(k_p)}(t), t)$, $t_0 \leqslant t \leqslant t_0 + T$.

---

1) By the symbol $\operatorname{mes} E(|u_j^{(1)}(t) - u_j^{(k)}(t)| \geqslant \sigma)$ is meant the measure of the set in which the inequality $|u_j^{(1)}(t) - u_j^{(k)}(t)| \geqslant \sigma$ is satisfied, σ -- any positive number.

Proof. From lemma 1.) and the inequality $|u_j^{(k)}(t)| \leqslant N$, $j = 1, 2, \ldots$, it follows that the sequence $x^{(k)}(t)$ is bounded, uniform in k, $t_0 \leqslant t \leqslant t_0 + T$. Since the functions $f(x, t)$ $b_{ij}(t)$, $u^{(k)}(t)$ are bounded in t (uniform in k), then from the inequality

$$|x_i^{(k)}(t) - x_i^{(k)}(t')| \leqslant \int_{t'}^{t} \left| f_i(x^{k}, t) + \sum_{j=1}^{r} b_{ij}(s) u_j^{(k)}(s) \right| ds$$

follows the equi-degree continuity of the functions $x^{(k)}(t)$.

Thus, the functions $x^{(k)}(t)$, $k = 1, 2, \ldots$, $t_0 \leqslant t \leqslant t_0 + T$ are uniformly bounded and equi-degree continuous. Consequently [6], there exists at least one uniformly convergent subsequence $x^{(k_l)}(t)$ of the sequence $x^{(k)}(t)$, $t_0 \leqslant t \leqslant t_0 + T$. Subsequently, we will consider what limits the generality of reasoning: if $u^{(k)}(t)$ -- minimizing sequence of the control vectors, then the corresponding sequence of trajectories of equation (1.1) $x(x_0, t_0, u^{(k)}(t), t)$ converge uniformly to some continuous function $x^0(t)$, $t_0 \leqslant t \leqslant t_0 + T$.

We will examine the equation of perturbed motion for (1.1) with the variation of the control $\delta u(t)$. As is known ([7], p 296), it has the form

$$\frac{d\delta x(t)}{dt} = \sum_{i=1}^{n} \frac{\partial f}{\partial x_i} \delta x_i + B(t)\, \delta u(t) + \theta(\delta x, t),$$

where the function $\theta(\delta x, t)$ satisfies the relationships

$$\lim \frac{\theta(\delta x, t)}{\sum_{i=1}^{n} |\delta x_i|} \to 0 \quad \text{for} \quad \sum_{i=1}^{n} |\delta x_i| \to 0.$$

Let $u^{(k)}(t)$ -- minimizing sequence of the control vectors of equation (1.1). Then, as was shown above, $x^{(k)}(t) \to x^0(t)$ for $k \to \infty$, $t_0 \leqslant t \leqslant t_0 + T$. If the matrix function $\frac{\partial f_i}{\partial x_j}$ is computed along the curves $x^0(t)$, $x^{(k)}(t)$, $k = 1, 2,$

..., then the equation of perturbed motion can subsequently be written as:

$$\frac{d\tilde{\delta}x}{dt} = P^{(k)}(t)\tilde{\delta}x + B(t)\delta u(t) + \theta(\tilde{\delta}x, t), \quad k = 0, 1, \ldots \qquad (1.3)$$

But the linear approximation equation corresponding to equation (1.3), we will write in the form

$$\frac{d\delta x}{dt} = P^{(k)}(t)\delta x + B(t)\delta u(t), \quad k = 0, 1, \ldots \qquad (1.4)$$

If $\delta u(t) = \delta u^{(k)}(t)$, then for the corresponding solutions $\tilde{\delta}x^{(k)}(t)$, $\delta x^{(k)}(t)$ of equations (1.3) and (1.4) the following is true

Lemma 1.3. The solutions of the systems (1.3) and (1.4) satisfy the inequality

$$|\tilde{\delta}x_i^{(k)}(t) - \delta x_i^{(k)}(t)| \leq \Phi(\varepsilon), \quad k = 1, 2, \ldots$$

where $t_0 \leq t \leq \tau$,

$$\varepsilon = \int_{t_0}^{\tau} \sum_{j=1}^{r} |\delta u_j^{(k)}(t)|\, dt \text{ and } \frac{\Phi(\varepsilon)}{\varepsilon} \to 0 \quad \text{for } \varepsilon \to 0.$$

Proof. On the grounds of lemma 1.1 we have

$$\sum_{i=1}^{n} |\tilde{\delta}x_i^{(k)}(t)| \to 0,$$

if

$$\operatorname{mes} E(|\delta u_j^{(k)}(t)| > \varepsilon) \to 0, \quad \varepsilon > 0, \quad j = 1, \ldots, r.$$

Seeing that the matrix function is continuous according to $\frac{\partial f_i}{\partial x_i}$ condition, then the function $\theta(\tilde{\delta}x^{(k)}, t)$ satisfies the Lipschitz conditions

$$|\theta_i(\tilde{\delta}x^{(k)}, t)| \leq l \sum_{i=1}^{n} |\tilde{\delta}x_i^{(k)}(t)|,$$

where l -- constant, as small as one wants for sufficiently small disturbances $\delta x^{(k)}(t)$. Supposing $\varepsilon = \int_{t_0} \sum_{j=1} |\delta u_j^{(k)}(t)| dt$

and recalling the above, we have

$$|\theta_i(\overline{\delta}x^{(k)}, t)| \leqslant R(\varepsilon), \quad \frac{R(\varepsilon)}{\varepsilon} \to 0 \text{ for } \varepsilon \to 0.$$

If we subtract equation (1.4) from (1.3) and integrate through, considering the correct inequality

$$|\theta_i(\overline{\delta}x^{(k)}, t)| < R(\varepsilon). \quad \frac{R(\varepsilon)}{\varepsilon} \to 0 \text{ for } \varepsilon \to 0.$$

we finally obtain

$$|\overline{\delta}x_i^{(k)}(t) - \delta x_i^{(k)}(t)| \leqslant \Phi(\varepsilon).$$

where $\frac{\Phi(\varepsilon)}{\varepsilon} \to 0$ for $\varepsilon \to 0$, while the value is uniform in k.

§2. We will examine the equation

$$\frac{dz(t)}{dt} = P^{(k)}(t)\, z(t) + B(t)\, w(t). \qquad k = 0, 1, \ldots \qquad (2.1)$$

where the elements $P_{ij}^{(k)}(t)$ of the matrix $P^{(k)}(t)$ are equal to the functions $\frac{\partial f_i}{\partial x_j}$, calculated along the curves $x^{(k)}(t)$, $k = 0, 1, \ldots$, which satisfy the relation $x^{(k)}(t) \to x^0(t)$ uniformly for $k \to \infty$.

We will introduce notations if it is necessary to examine the j-column of the matrix $B(t)$ then we will write it as: $B_j(t)$. For a scalar derivative of the vectors l and $F_{(k)}^{-1}(t)\, B_j(t)$, where $F_{(k)}^{-1}(t)$ - matrix, inverse to the fundamental matrix of equation (2.1) with $B(t) \equiv 0$, we will use the symbol: $(l \cdot [F_{(k)}^{-1}(t)\, B_j(t)])$.

We suppose that the relations

$$(l\cdot[F_{(k)}^{-1}(t)B_j(t)])=0,\quad \sum_{i=1}^{n} l_i^2\neq 0,\qquad j=1,\ldots,r,\ k=0,1,\ldots, \tag{2.2}$$

hold only at distinct, isolated points. As is known [4], if a control vector $w(t)$ is such that a point moving from z_0 along the trajectory of equation (2.1) encounters the origin of the coordinates at moment of time τ, then the vector $w(t)$ is a solution of the system

$$-z_0=\int_{t_0}^{\tau} F_{(k)}^{-1}(t)B(t)\,w(t)\,dt. \tag{2.3}$$

Here none of the numbers z_0 are connected with the original problem.

Now we will examine the minimizing sequence of the controls $u^{(k)}(t)$. The control time for the trajectories which correspond to them equals T_k and $T_k \to T$ for $k \to \infty$, T_k -- decreasing sequence. Consequently, there exists a sequence τ_k, which satisfies the inequalities $T_k - \tau_k < T$ and the relationship $\tau_k \to 0$ for $k \to \infty$. We suppose

$$-\int_{t_0}^{t_0+T_k-\tau_k} F_{(k)}^{-1}(t)B(t)\,u^{(k)}(t)\,dt=z_0^{(k)},\qquad k=1,2\ldots \tag{2.4}$$

It is clear that of the points $z_0^{(k)}$, $k = 1, 2, \ldots$, along the trajectory of the equation (2.1) the origin of the coordinates is accessible (even with the control $u^{(k)}(t)$). If by the symbol $O(T_k - \tau_k)$ is understood the set of points of phase space z, for which the origin of the coordinates is accessible in time $t \leqslant T_k - \tau_k$ along the trajectory of the equation (2.1) then, it is apparent that the points $z_0^{(k)}$, $k = 1, 2, \ldots$, belong to the corresponding regions $O(T_k - \tau_k)$.

From the results of the work [8] it follows that each of the regions $O(T_k - \tau_k)$, $k = 1, 2, \ldots$, is convex and closed. We construct lines from the origin of the coordinates to the intersection with the boundaries of the regions $O(T_k - \tau_k)$, $k = 1, 2, \ldots$, and we designate the greatest distance between corresponding points of the boundaries of

the regions $G(T_k - \tau_k)$ and $G(T_{k+1} - \tau_{k+1})$, $k = 1, 2, \ldots$, obtained by the result of such a construction by d_k. Then it can be proved, considering the results of article [9], that $d_k \to 0$ for $k \to \infty$.

Subsequently we will speak about the fact that the sequences of regions $G(T_k - \tau_k)$ converge to the region $G(T)$. In the present case, the limit region $G(T)$ consists of points, for which the origin of the coordinates is accessible in time $t \leqslant T$ along the trajectory of equation (2.1) for $k = 0$, i.e., when the functions $\frac{\partial f_i}{\partial x_l}$ are calculated along the curve $x^0(\tau)$, and is, as can be shown, considering the supposition (2.2), a non-empty set. We will designate the boundaries of the regions $G(T_k - \tau_k)$ by Γ_k respectively. The following assertion is proved:

Lemma 2.1. All limit points of the sequence $z_0^{(k)}$ belong to the boundary Γ of the region $G(\Gamma)$.

Proof. In view of the above mentioned, the sequence of regions $G(T_k - \tau_k)$ is uniformly bounded in k. Consequently the set of points $z_0^{(k)}$ has at least one limit point z_0. We shall prove that z_0 belongs to Γ.

Assume the contrary. Then, in light of the fact that the sequence of regions $G(T_k - \tau_k)$ converges to $G(T)$ for $k \to \infty$, there exists a subsequence $z_0^{(k)}$, every point of which can be surrounded by a sphere of radius ρ_{k_l} ($\rho_{k_l} \to \rho_0$ for $l \to \infty$), $\rho_{k_l} > \rho_0 \neq 0$, whereupon, all points of the spheres belong to the corresponding regions of the coordinate lines to the intersection with the boundary Γ_{k_l} of the region $\Gamma(T_{k_l} - \tau_{k_l})$.

Such an auxiliary construction for each point $\bar{z}^{(k_l)}$ of the surface of the sphere of radius ρ_{k_l} with center at the point $z_0^{(k_l)}$ puts it in correspondence with some point $\tilde{z}^{(k_l)}$ of the boundary Γ_{k_l}. If $\alpha(\bar{z}^{(k_l)})$ -- the ratio of the distance from the origin of the coordinates to some point of the surface of the sphere $\bar{z}^{(k_l)}$ (radius of the sphere ρ_{k_l},

center at the point $z_0^{(k_l)}$ to the distance from the origin of the coordinates to the corresponding point $\tilde{z}^{(k_l)}$ of the surface Γ_{k_l}, then it is not difficult to see that the function $\alpha(\bar{z}^{(k_l)})$ satisfies the inequality $0 < \alpha(\bar{z}^{(k_l)}) < 1$ and is continuous in $\bar{z}^{(k_l)}$.

As is known [3-4], for each point $\tilde{z}^{(k_l)}$ there exists a unique optimal control $\tilde{w}^{(k_l)}$, which is a piecewise constant function. From equation (2.3) it follows that the vector $\alpha(\bar{z}^{(k_l)})\tilde{w}^{(k_l)}$ is a controlling one for the point $\bar{z}^{(k_l)}$. Thus, for each point of the surface of the sphere $\bar{z}^{(k_l)}$ a controlling vector takes a function $\alpha(\bar{z}^{(k_l)})\tilde{w}^{(k_l)}$. From the property of continuity of the function $\tilde{w}^{(k_l)}$ from the initial values and from the continuity of $\alpha(\bar{z}^{(k_l)})$ in $\bar{z}^{(k_l)}$ we obtain that the function $\alpha(\bar{z}^{(k_l)})\tilde{w}^{(k_l)}(t)$ also is continuous in $\bar{z}^{(k_l)}$.1)

We will examine the function

$$w^{(k_l)}(t) = u^{(k_l)}(t) + \lambda\,[\alpha(\bar{z}^{(k_l)})\,\tilde{w}^{(k_l)}(t) - u^{(k_l)}(t)], \tag{2.5}$$

where λ satisfies the inequality $0 \leqslant \lambda \leqslant 1$. Seeing that the vector $u^{(k_l)}(t)$ is a controlling one for the initial conditions $z_0^{(k_l)}$, then for the substitution of λ into the mentioned limits, formula (2.5) gives a control for each interior point of the sphere of radius ρ_{k_l} with center at $z_0^{(k_l)}$, and for $\lambda = 1$ we obtain controlling vectors for the points $\bar{z}^{(k_l)}$ of the surface of the sphere.

It is clear that the function $w^{(k_l)}(t)$ is continuous in the initial values and satisfies the inequality

1) We remember that we call $\tilde{w}^{k_l}(z, t)$ continuous, according to the initial values z, if for each $\varepsilon > 0$ there exists some $\delta > 0$, that the inequality

$$\operatorname{mes} F(|\tilde{w}_1^{(k_l)}(z_1, t) - \tilde{w}^{(k_l)}(z_2, t)| > \varepsilon) < \varepsilon, \quad \varepsilon > 0.$$

is fulfilled only if

$$|z_{l_1} - z_{l_2}| < \delta.$$

$$|w_j^{(k_l)}(t)| \leqslant N.$$

We put

$$\delta u^{(k_l)}(t) = \lambda\{e(\widetilde{z}^{k_l})\,\widehat{w}^{k_l}(t) - u^{(k_l)}(t)\} \tag{2.6}$$

and examine the linear approximating equation (1.4) with the control (2.6).

Let $\delta x^{(k_l)}(t_0) = 0$. Then the variations $\delta x^{(k_l)}(t)$ are found by the formulas

$$\delta x^{(k_l)}(t) = F_{(k_l)}(t)\int_{t_0}^{t} F_{(k_l)}^{-1}(\tau)\,B(\tau)\,\delta u^{(k_l)}(\tau)\,d\tau$$

or

$$F_{(k_l)}^{-1}(t)\,\delta x^{(k_l)}(t) = \int_{t_0}^{t} F_{(k_l)}^{-1}(\tau)\,B(\tau)\,\delta u^{(k_l)}(\tau)\,d\tau. \tag{2.7}$$

With the variation $\delta u^{(k_l)}(t)$, formulas (2.6) and (2.5) agree that in space z at the moment of time $t = t_0 + T_{k_l} - \tau_{k_l}$ the origin of the coordinates is reached from each point of the sphere of radius ρ_{k_l}, then it follows from (2.7) that the points of the trajectories $x^{(k_l)}(t) + \delta x^{(k_l)}(t)$ of the space x at the moment $t = t_0 + T_{k_l} - \tau_{k_l}$ form an ellipse by a nonsingular transformation $F_{(k_l)}(t_0 + T_{k_l} - \tau_{k_l})$ of the sphere.

Seeing that $F_{(k_l)}(t_0 + T_{k_l} - \tau_{k_l}) \to F_0(T)$ for $l \to \infty$, then the sequence of the ellipsoids, defined by (2.7), has for its limit an ellipsoid, which can be found from the circle of radius ρ_0 by a dependent transformation $F_0(T)$. But seeing that $T_{k_l} \to \tau_{k_l} \to T$ for $l \to \infty$ we conclude: there exists some number l_1, beginning with which all the ellipsoids $(l > l_1)$ contain the origin of the coordinates.

We return to equation (2.1). From the continuity of the controlling vector $w^{(k_l)}(t)$ (see (2.5)) the initial values $z^{(k_l)}$ and from formula (2.4) it follows that if we decrease the controls $\delta u_j^{(k_l)}(t)$, $j = 1, 2, \ldots, r$, in ε time, $0 < \varepsilon < 1$, then the controls $u^{(k_l)}(t) + \varepsilon\delta u^{(k_l)}(t)$ will correspond to the initial conditions $z^{(k_l)} + \varepsilon\delta z^{(kl)}$

$$(\delta z^{(k_l)} = z^{(k_l)} - z_0^{(k_l)}).$$

Consequently, for the decrease of the controls of $\delta u^{(k_l)}(t)$ in time $\varepsilon < 1$, the radii of the spheres decrease in time ε.

We designate a radius-vector coming from the center $x^{(k_l)}(t_0 + T_{k_l} - \tau_{k_l})$ of an ellipsoid defined by equation (2.7) by η_{k_l}; then, obviously, for a decrease in the radius of a sphere ρ_{k_l} in time ε the magnitude of the radius-vector becomes equal to $\varepsilon\eta_{k_l}$.

But all ellipsoids for $l > l_1$ contain the origin of the coordinates. Applying lemma 1.3 we obtain $\max|\tilde{\delta x}_l^{(k_l)}(t) - \delta x_l^{(k_l)}(t)| \leqslant a_{k_l} - \mu_{k_l}$, where a_{k_l} is equal to the semi-minor axis of the ellipse, and μ_{k_l} is equal to the distance from the origin to the point $x^{(k_l)}(t_0 + T_{k_l} - \tau_{k_l})$, $\varepsilon > \varepsilon$, $l > l_3$.

Thus, for the variation $\tilde{\delta x}^{(k_l)}(\tau)$, when the controlling vectors are transformed according to the formula $\omega\lambda[u(z_0^{(k_l)})\tilde{w}^{(k_l)}(t) - u^{(k_l)}(t)]$, the points of the trajectories $x^{(k_l)}(t) + \tilde{\delta x}^{(k_l)}(t)$ in a moment of time $t = t_0 + T_{k_l} - \tau_{k_l}$ comprise a continuous set Q_{k_l}, agreeing with lemma 1.1. In view of the above mentioned and lemma 1.1, we arrive at the following:

The ellipsoids of the radius-vector $\varepsilon\eta_{k_l}$ for $\varepsilon < \varepsilon_1$ and $l > l_3$, which contain the origin, are continuously mapped into continuous sets Q_{k_l}, which also contain the origin.

Consequently, we find in the conditions of applicability of the theorem on the existence of a root ([10], p 573), that in our case it corresponds to the existence of a trajectory for equation (1.1), along which the origin is reached from the point x_0 in time $t = T_{k_l} - \tau_{k_l} < T$. But this is impossible according to the condition of the problem.

It means $\rho_{k_l} \to 0$ for $l \to \infty$. In other words, the point

z_0 belongs to the boundary Γ of the region $G(T)$. The lemma is proved.

Insofar as the point z_0 belongs to the boundary of the region $G(T)$, then the time T is the optimal control time for the initial values z_0, and the optimal controlling vector is found from the relation

$$u_j^0(t) = N \operatorname{sign}(l^0 \cdot [F_0^{-1}(t) B_j(t)]), \quad (l^0 \cdot z_0) = -1, \quad j = 2, \ldots, r. \tag{2.8}$$

where the vector l^0 is obtained by the condition

$$\min \int_{t_0}^{t_0+T} \sum_{j=1}^{r} |(l \cdot [F_0^{-1}(t) B_j(t)])| \, dt, \quad (l \cdot z^0) = -1.$$

§3. Now we shall prove on the basis of the results of §1 and §2 the following theorem.

Theorem. If for equation (1.1) for some control $u^{(1)}(t)$ the origin of the coordinates is accessible in time T_1, and the linear approximating equation satisfies the conditions (2.2) in the region $|x_i(t)| \leqslant nr^2 MN T_1 e^{nLT_1}$ ($M = \max|b_{ij}(t)|$, $t_0 \leqslant t \leqslant t_0 + T_1$), then there exists at least one optimal control which is a piecewise constant function and is defined by the formulas (2.8).

Proof.[1] If the controlling vector $u^{(1)}(t)$ is not optimal, then there exists a control $u^{(2)}(t)$, for which the origin is accessible from point x_0 in time $t_2 < T_1$.

By reasoning similarly, we will convince ourselves of the existence of a minimizing sequence of controlling vectors $u^{(k)}(t)$, $k = 1, 2, \ldots$, for which the control time is respectively equal to T, $T_{k+1} < T_k$, $\lim T_k = T > 0$ for $k \to \infty$.

The inequality $T > 0$ follows from the fact, that the right parts of equations (1.1) are bounded. It can occur that the number of controlling vectors in a minimizing sequence is finite; this corresponds to the case, when for

[1] The theorem can also be proved by application of L. S. Pontryagin's principle of the maximum.

equation (1.1), besides the control $u^{(1)}(t)$, there exists a finite number of controls, for which the origin is accessible in time $t < T_1$. In such a case we can immediately speak of the existence of an optimal vector $u(t)$.

The vector $u(t)$ is computed by the formula (2.8). This fact will be demonstrated below for the more general case, namely, when the members of the minimizing sequence are distinct.

Thus, let $u^{(k)}(t)$ be a minimizing sequence of controlling vectors of the control (1.1). As was shown in the basis of the lemma 1.2, the sequences of trajectories $x^k(\tau) = x(x_0, t_0, u^{(k)}(t), \tau)$ uniformly converges to $x^0(t)$ for $k \to \infty$. If

$$|x_i(t)| \leqslant nr^3MN\, T_1\, e^{nL}\, T_1,$$

where

$$M = \max |b_{ij}(t)|, \quad t_0 \leqslant t \leqslant t_0 + T_1,$$

(see lemma 1.1), then we can construct a sequence $z_0^{(k)}$, which is defined by formulas (2.4). In the basis of lemma 2.1, the sequence $z_0^{(k)}$ has at least one limit point z_0, which belongs to the boundary $G(T)$. Let $z_0^{(k_l)} \to z_0$ for $l \to \infty$. We obtain the difference

$$z_0 - z_0^{(k_l)} = \int_{t_0}^{t_0+T} [F_{(k_l)}^{-1}(t)\, B(t)\, u^{(k_l)}(t) - F_0^{-1}(t)\, B(t)\, u^0(t)]\, dt -$$

$$- \int_{t_0+T_{k_l}-\tau_{k_l}}^{t_0+T} F_{(k_l)}^{-1}(t)\, B(t)\, u^{(k_l)}(t)\, dt = - \int_{t_0}^{t_0+T} F_0^{-1}(t)\, B(t)\, [u^0(t) - u^{(k_l)}(t)]\, dt +$$

$$+ \int_{t_0}^{t_0+T} [F_{(k_l)}^{-1}(t) - F_0^{-1}(t)]\, B(t)\, u^{(k_l)}(t)\, dt -$$

$$- \int_{t_0+T_{k_l}-\tau_{k_l}}^{t_0+T} F_{(k_l)}^{-1}(t)\, B(t)\, u^{(k_l)}(t)\, dt.$$

We introduce the notation

$$-z_3 + z_0^{(k_l)} + \int_{t_0}^{t_0+T} [F_{(k_l)}^{-1}(t) - F_0^{-1}(t)]\, B(t)\, u^{(k_l)}(t)\, dt -$$

$$- \int_{t_0+T_{k_l}-\tau_{k_l}}^{t_0+T} F_{(k_l)}^{-1}(t)\, B(t)\, u^{(k_l)}(t)\, dt = c_{k_l}.$$

It is clear that $c_{k_l} \to 0$ for $e \to \infty$. But

$$c_{k_l} = \int_{t_0}^{t_0+T} F_0^{-1}(t)\ B(t)\, [u^0(t) - u^{(k_l)}(t)]\, dt.$$

We scalar multiply the vector l^0 (see (2.8)) by c_{k_l}

$$(l^0 \cdot c_{k_l}) = \int_{t_0}^{t_0+T} \sum_{j=1}^{r} (l^0 \cdot [F_0^{-1}(t)\, B_j(t)])\, [u_j^0(t) - u_j^{(k_l)}(t)]\, dt.$$

But $u_j^0(t) = N \operatorname{sign}(l^0 [F_0^{-1}(t)\, B_j(t)])$, $(l^0 \bar{z}_0) = -1$ and by the condition of the problem $|u_j^{(k_l)}(t)| \leqslant N$, consequently,

$$\operatorname{sign}(l^0 [F_0^{-1}(t)\ B_j\ (t)]) = \operatorname{sign}(u_j^0\ (t) - u_j^{(k_l)}(t)).$$

Thus, there is obtained

$$(l^0 c_{k_l}) = \int_{t_0}^{t_0+T} \sum_{j=1}^{r} |(l_0 [F_0^{-1}(t)\, B_j(t)])|\ |u_j^0(t) - u_j^{(k_l)}(t)|\, dt.$$

From $(l^0 c_{k_l}) \to 0$ for $e \to \infty$ and condition (2.2), follows that $\operatorname{mes} E(|u_j^0(t) - u_j^{(k_l)}(t)| \geqslant \sigma) \to 0$ for $e \to \infty$ and for each $j = 1, 2, \ldots, r$. That means the sequence $u^{(k_l)}(t)$ converges by measure to the function $u^0(t)$. It is not difficult to see that the vector $u^0(t)$ is optimal for the control (1.1).

Actually, seeing that the functions $x^{(k_l)}(t)$ are solutions of the equations (1.1) there are the relations

$$x^{(k_l)}(t) = x_0 + \int_{t_0}^{t} [f(x^{(k_l)}, \tau) + B(\tau) u^{(k_l)}(\tau)] d\tau.$$

But $x^{(k_l)}(t) \to x^0(t)$ uniformly for $e \to \infty$, $u^{(k_l)}(t) \to u^0(t)$ in measure, the function $f(x, t)$ is continuous x, consequently [11], can in the limit be put under the integral sign. We obtain

$$x^0(t) = x_0 + \int_{t_0}^{t} [f(x^0, \tau) + B(\tau) u^0(\tau)] d\tau.$$

Thus, the trajectory $x^0(t)$ corresponds to the control $u^0(t)$ of equation (1.1). The equality $x(x_0, t_0, u^0(t), t_0 + T) = 0$ immediately follows from the continuity of the functions $x^k(t)$ and $x^0(t)$ in t. We at the same time proved that the optimal controlling vector $u^0(t)$ is computed according to formula (2.8), i.e., it is a piecewise constant function. The theorem is proved.

The restrictions (2.2) imposed upon the linear approximation equations, can be expressed in a form, which in a series of cases permits in view of (1.1) judging the existence of a solution of the problem of optimal control with a piecewise constant controlling vector of the type (2.8). Such conditions for a linear equation with constant coefficients are given in work [3]. Sufficient conditions will be introduced below for the fulfillment of (2.2) in the case of a linear equation ($n \leqslant 3$).

We will examine the linear approximating equation for (1.1), with the corresponding variations $\delta u(t)$, when the matrix function $\frac{\partial f_i}{\partial x_l}$ is calculated along some curve x(t)

$$\frac{d\delta x}{dt} = P(t)\delta x(t) + B(t)\delta u(t).$$

It will also be proved that the assertion is also true if the vectors, $B_j(t)$, $R_j^{(1)}(t)$, $\frac{dR_j^{(1)}(t)}{dt} - P(t)R_j^{(1)}(t)$, where $R_j^{(1)}(t) = \frac{dB_j(t)}{dt} - P(t)B_j(t)$ for all j and x(t) arising from the equa-

tions (1.2), are noncollinear, then the relation (2.2) is true. Actually, we will examine the vector $H_j(t) = F^{-1}(t)$ $B_j(t)$, seeing that $\frac{dF^{-1}(t)}{dt} = -F^{-1}(t)\,P(t)$ [7], then

$$H_j'(t) = F^{-1}(t)\left[\frac{dB_j(t)}{dt} - P(t)B_j(t)\right].$$

We call $\frac{dB_j(t)}{dt} - P(t)B_j(t)$ by $R_j^1(t)$, then we will obtain a formula for the determination of the second derivative $H_j''(t)$

$$H_j''(t) = F^{-1}(t)\left[\frac{dR_j^1(t)}{dt} - P(t)\,R_j^1(t)\right].$$

Let the relation $(l\cdot[F^{-1}(t)B_j(t)]) = 0$ hold in a set, distinct from a collection of isolated points. Then the relations $(lH_j(t)) = 0$, $(lH_j'(t)) = 0$, and $(l\cdot H_j''(t)) = 0$ hold in the same set. But this is impossible, seeing that according to condition the vectors

$$B_j(t),\ R_j^{(1)}(t),\ \frac{dR_j^{(1)}(t)}{dt} - P(t)\,R_j^{(1)}(t), \quad j = 1, 2, \ldots, r,$$

are non-collinear and the matrix $F^{-1}(t)$ is non-singular, what was required to prove.

Let the vectors

$$B_j(t),\ R_j^{(1)}(t) = \frac{dB_j(t)}{dt} - P(t)B_j(t),\ \frac{dR_j^{(1)}(t)}{dt} - P(t)R_j^{(1)}(t)$$

be non-collinear. We will subsequently call such a fact condition (A). Then the theorem of existence of optimal trajectories can be expressed as: if for equation (1.1) for some control $u^{(1)}(t)$ the origin is accessible for $t = T_1$, and the linear approximating equations satisfy the conditions (A) in the region

$$|x_i(t)| \leqslant nr^2MN(T_1 - t_0)e^{nL(T_1-t_0)},$$

then there exists at least one optimal equation which is a piecewise constant function and is defined by formula (2.8). We shall write conditions (A) in clear form for n = 2, 3. For n = 2 we have

$$R_j^{(1)}(t) = \left(\frac{db_{1j}(t)}{dt} - \sum_{i=1}^{2} \frac{\partial f_1}{\partial x_i} b_{ij}(t), \ \frac{db_{2j}(t)}{dt} - \sum_{i=1}^{2} \frac{\partial f_2}{\partial x_i} b_{ij}(t) \right).$$

By the symbol $[R_j^{(1)}(t)]_k$ is meant the k-component of the vector $R_j^{(1)}(t)$, the entry $[R_j^{(1)}(t)]_k$, k = 1, 2, ..., n, means the vector $R_j^{(1)}(t)$.

Then condition (A) for n = 2 reduces to the non-collinearity of the vectors (b_{1j}, b_{2j}), $[R_j^{(1)}(t)]_k$, k = 1, 2. For n = 3

$$[R_j^{(1)}(t)]_k = \frac{db_{kj}(t)}{dt} - \sum_{i=1}^{3} \frac{\partial f_k}{\partial x_i} b_{ij}(t), \quad k = 1, 2, 3.$$

We define the vector $R_j^{(2)}(t) = \frac{dR_j^{(1)}(t)}{dt} - P(t) R_j^{(1)}(t)$. Seeing that

$$R_j^{(1)}(t) = \frac{dB_j(t)}{dt} - P(t) B_j(t),$$

then

$$R_j^{(2)}(t) = \frac{d^2 B_j(t)}{dt} - \frac{dP(t)}{dt} B_j(t) - 2P(t) \frac{dB_j(t)}{dt} + P^2(t) B_j(t),$$

where $p_{ij}(t) = \frac{\partial f_i}{\partial x_j}$, means,

$$\frac{dp_{ij}(t)}{dt} = \sum_{l=1}^{3} \frac{\partial^2 f_i}{\partial x_j \partial x_l} \frac{dx_l}{dt} + \frac{\partial^2 f_i}{\partial x_j dt}, \qquad p_{ij}^2(t) = \sum_{l=1}^{3} \frac{\partial f_i \partial f_l}{\partial x_l \partial x_j},$$

and, consequently, we finally have

$$[R_j^{(2)}(t)]_k = \frac{d^2 b_{jk}(t)}{dt^2} - \sum_{i=1}^{3}\sum_{l=1}^{3}\left\{\frac{\partial^2 f_k}{\partial x_i \partial x_l}\left[f_l(x, t) + \sum_{m=1}^{r} b_{lm}(t)u_m(t)\right] + \right.$$

$$\left. + \frac{\partial^2 f_k}{\partial x_i dt} b_{jk}(t) + 2\frac{\partial f_k}{\partial x_i}\frac{db_{ik}(t)}{dt} - \frac{\partial f_k}{\partial x_l}\frac{\partial f_l}{\partial x_i} b_{ik}(t)\right\}.$$

Thus, if the vectors (b_{1j}, b_{2j}, b_{3j}), $[R_j^{(1)}(t)]_k$, $k = 1, 2, 3$, $j = 1, \ldots, r$, are non-collinear, then condition (2.2) for the linear approximating equation will hold, while in this case (for $n = 3$) for verification it is sufficient for (2.2) condition and necessary to compute the derivatives up until the second order exclusively from the functions $b_{ij}(t)$, $f(x, t)$ in x and t.

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